From blind source separation to tensor decomposition: an algebraic algorithm

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joint work with Ignat Domanov and Lieven De Lathauwer

Numa seminar
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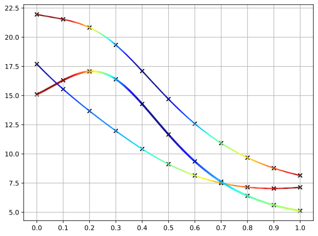
KU LEUVEN kulak
STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics
erc European Research Council Established by the European Commission

Overview

Blind Source Separation problem



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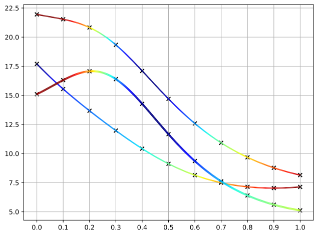

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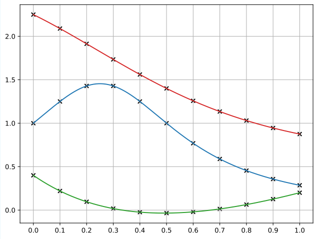

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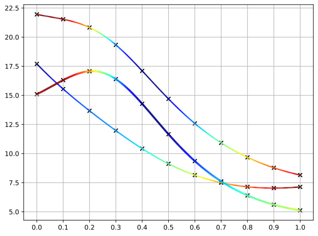

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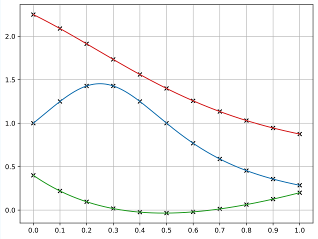

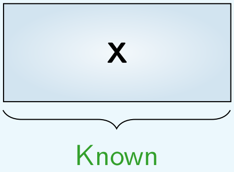

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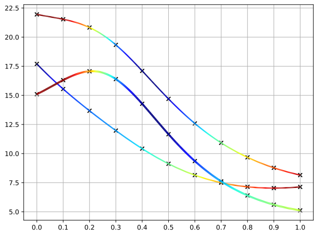

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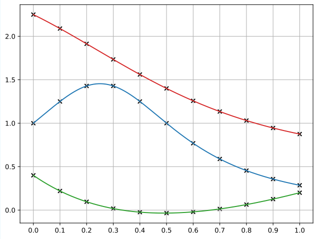

$$\underbrace{\mathbf{X}}_{\text{Known}} = \underbrace{\mathbf{M}}_{\text{Unknown}} \; \underbrace{\mathbf{S}^T}_{\text{Unknown}}$$

Overview

Constraints for uniqueness

Let $\mathbf{X}$ be a matrix

Definition: a *deterministic condition of* $\mathbf{X}$ is a particular matrix property which is always true.

Definition: a *generic condition of* $\mathbf{X}$ depends on a parameter $\mathbf{z} \in \Omega$ and holds almost everywhere, i.e., if Σ is the set of $\mathbf{z}$ values for which the condition doesn't hold, then $\mu(\Sigma) = 0$ where μ is a measure absolute continuous w.r.t. the Lebesgue one.

Deterministic conditions

- Statistical independence → Independent Component Analysis;

$$\mathbf{X} = \mathbf{M} \underset{\text{Ind}}{\mathbf{S}^T}$$

[Comon and Jutten 2009]

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- ...

[Comon and Jutten 2009]

General case

$$\mathbf{X} = \mathbf{M} \quad \mathbf{S}^T$$

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$$\underbrace{\mathbf{X}}_{\mathbf{X}} = \underbrace{\mathbf{M} \; \mathbf{G}}_{\mathbf{M}_1} \; \underbrace{\mathbf{G}^{-1} \; \mathbf{S}^T}_{\mathbf{S}_1^T}$$

Uniqueness isn't guaranteed!

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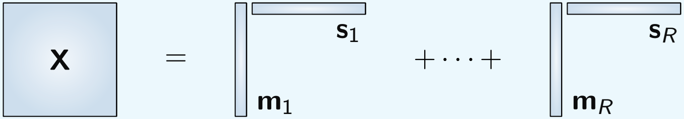
Problem statement



$$\mathbf{X} = \sum_{r=1}^{R} \mathbf{m}_r(\mathbf{z}) \otimes \mathbf{s}_r(\mathbf{z})$$

where

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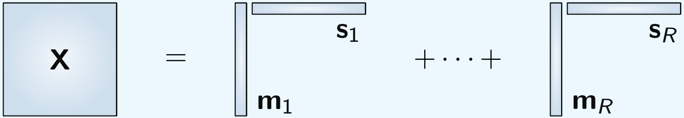

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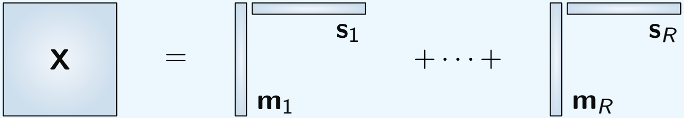

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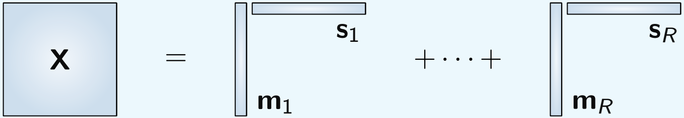

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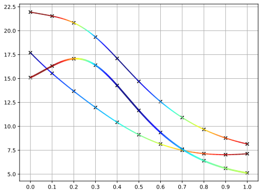
- each $\mathbf{s}_r(\mathbf{z})$ depends on ℓ independent parameters, entries of $\boldsymbol{\xi}_r \in \mathbb{R}^\ell$;
- each $\mathbf{s}_r(\boldsymbol{\xi}_r)$ has the structure

$$\mathbf{s}(\boldsymbol{\xi}_r) = \left[\frac{p_1}{q_1} \circ \mathbf{f}(\boldsymbol{\xi}_r) \quad \ldots \quad \frac{p_N}{q_N} \circ \mathbf{f}(\boldsymbol{\xi}_r) \right]$$

with p_h, q_h polynomials and $\mathbf{f} = [f_1, \ldots, f_\ell]$ a vectorial function.

Example



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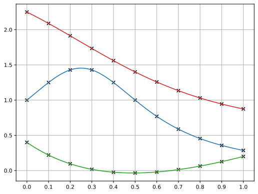

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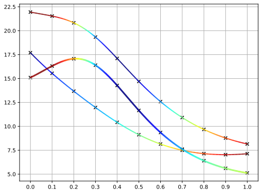

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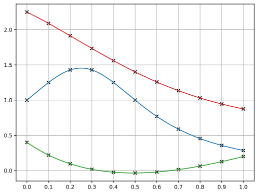

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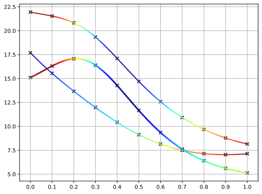

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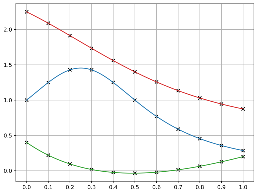

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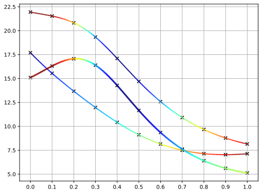

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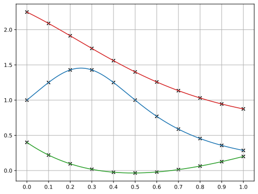

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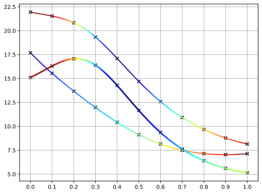

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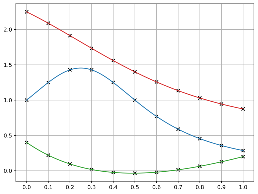

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- $\xi = [a_0, \ldots, a_p, b_0, \ldots, b_q]$
- $\ell = p + q + 2$
- $\mathbf{f}$ is the identity

A cookbook recipe for generic uniqueness [Domanov and De Lathauwer 2016]

Let $\mathbf{t}(\mathbf{x}) = \left[\frac{p_1}{q_1}(\mathbf{x}) \quad \dots \quad \frac{p_N}{q_N}(\mathbf{x})\right]^T$ for $\mathbf{x} \in \Theta = \{\mathbf{x} \in \mathbb{C}^\ell : q_1(\mathbf{x}) \cdots q_N(\mathbf{x}) \neq 0\}$, if

1. $\text{rank}\mathbf{M}(z) = R$ for a generic choice of $\mathbf{z}$;
2. each f_h is the ratio of two analytical functions on $\mathbb{C}^\ell$;
3. there exists $\boldsymbol{\xi}_0 \in \mathbb{C}^\ell$ s.t. $\det\mathbf{J}(\mathbf{f}, \boldsymbol{\xi}_0) \neq 0$;
4. the dimension of the span of $\mathbf{t}(\mathbf{x})$ for $x \in \Theta$ is at least $\hat{N}$;
5. $\text{rank}\mathbf{J}(\mathbf{t}, \mathbf{x}) > \hat{\ell}$ for a generic choice of $\mathbf{z}$;
6. $R \leq \hat{N} - \hat{\ell}$;

then

$$\mathbf{X} = \sum_{r=1}^{R} \mathbf{m}_r(\mathbf{z}) \otimes \mathbf{s}(\boldsymbol{\xi}_r)$$

is generically unique.

Remarks for BSS

It is assumed that the columns of $\mathbf{S}$ are values of the rational function

$$\mathbf{t} : \mathbf{x} \to \begin{bmatrix} \frac{p_1}{q_1}(\mathbf{x}) & \dots & \frac{p_N}{q_N}(\mathbf{x}) \end{bmatrix}^T .$$

Composed with the function $\mathbf{f}$

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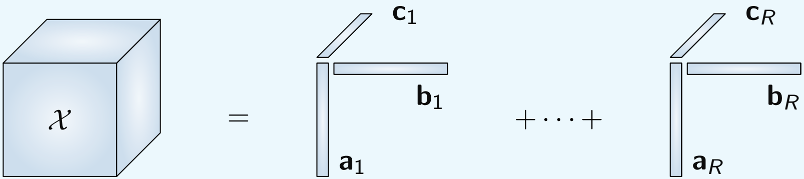
$$\mathcal{V} = \Big\{ (z_1, \ldots, z_N) \in \mathbb{C}^N : P_k(z_1, \ldots, z_N) = 0 \Big\}.$$

Composed with the function $\mathbf{f}$

Overview

Link with the CPD

$$\text{if } \mathcal{X} = \mathbf{a}_1 \otimes \mathbf{b}_1 \otimes \mathbf{c}_1 + \ldots + \mathbf{a}_R \otimes \mathbf{b}_R \otimes \mathbf{c}_R$$



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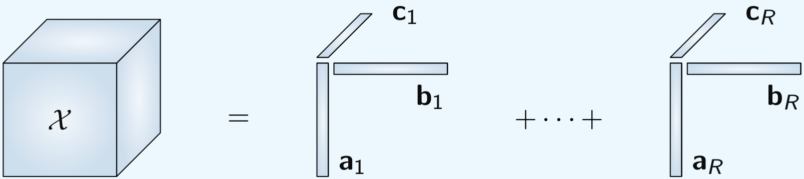
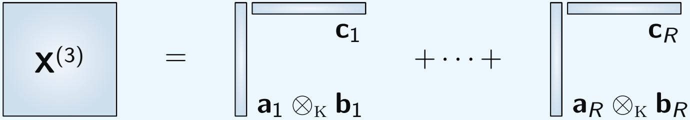


$$\text{then } \mathbf{X}^{(3)} = (\mathbf{a}_1 \otimes_{\mathrm{K}} \mathbf{b}_1) \otimes \mathbf{c}_1^T + \ldots + (\mathbf{a}_R \otimes_{\mathrm{K}} \mathbf{b}_R) \otimes \mathbf{c}_R^T$$



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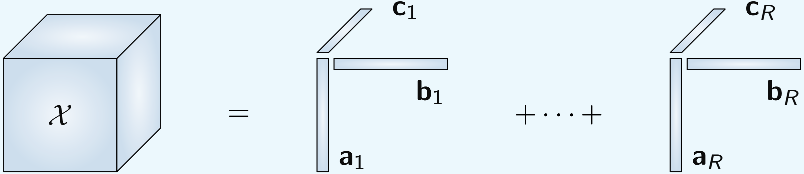

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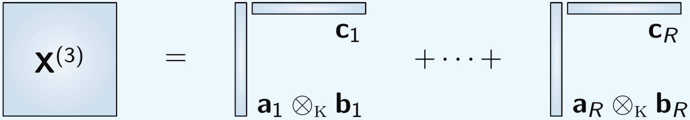

$$(\mathbf{a}_r \otimes_{\mathrm{K}} \mathbf{b}_r) \in \mathcal{V} = \left\{ \mathrm{vec}(\mathbf{Z}) : \begin{vmatrix} z_{i_1 j_1} & z_{i_1 j_2} \\ z_{i_2 j_1} & z_{i_2 j_2} \end{vmatrix} = 0 \right\}$$

Overview

Algebraic algorithm: high view

Let $\mathcal{X}$ be a $(I \times J \times R)$ tensor, then

$$\mathbf{X}^{(3)} = \sum_{r=1}^{R} (\mathbf{a}_r \otimes_{\mathrm{K}} \mathbf{b}_r) \otimes \mathbf{c}_r^T = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T.$$

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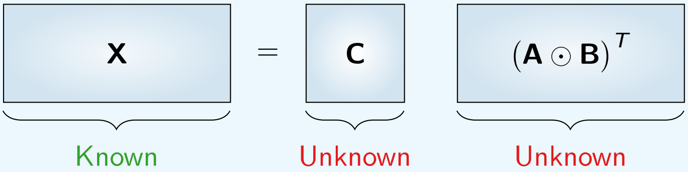

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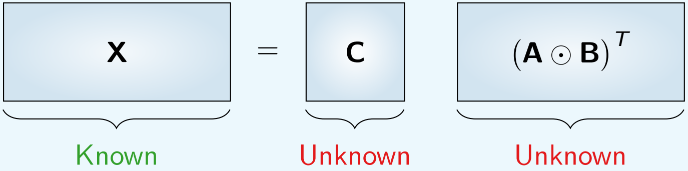

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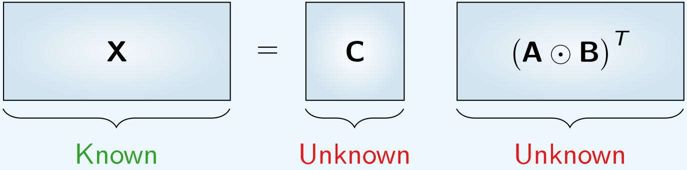

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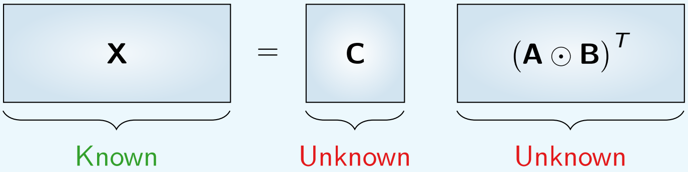

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3. factorize $(\mathbf{A} \odot \mathbf{B}) = [\mathbf{a}_1 \otimes_{\mathrm{K}} \mathbf{b}_1, \ldots, \mathbf{a}_R \otimes_{\mathrm{K}} \mathbf{b}_R]$ to recover $\mathbf{A}$ and $\mathbf{B}$;

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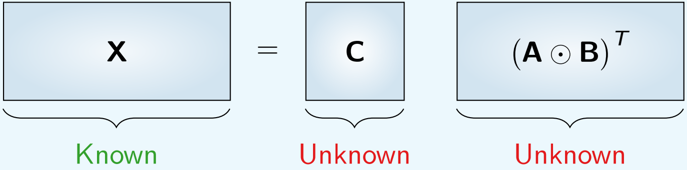

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3. factorize $(\mathbf{A} \odot \mathbf{B}) = [\mathbf{a}_1 \otimes_{\mathrm{K}} \mathbf{b}_1, \dots, \mathbf{a}_R \otimes_{\mathrm{K}} \mathbf{b}_R]$ to recover $\mathbf{A}$ and $\mathbf{B}$;
4. compute $\mathbf{C}$ by solving $(\mathbf{A} \odot \mathbf{B})\mathbf{C} = \mathbf{X}$.

Equivalent condition I

$\mathbf{c}$ is a column of $\mathbf{C}^{-1}$ if and only if $\mathbf{X}^T\mathbf{c}$ is equal to a column of $(\mathbf{A} \odot \mathbf{B})$

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where $P_k^{\otimes}(\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)$ is the vector obtained by formal substitution of $(z_1, \ldots, z_N)$ by $\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T$ and the scalar multiplication by the tensor product.

Equivalent condition II

$\mathbf{c}$ is a column of $\mathbf{C}^{-1}$ if and only if $\mathbf{X}^T\mathbf{c}$ is equal to a column of $\mathbf{A} \odot \mathbf{B}$

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$\Downarrow$

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The columns of $\mathbf{C}^{-1}$ belong to the intersection of $\mathbf{Q}$ kernel and $\mathrm{vec}(\mathrm{Sym}_R^N)$ the subspace of vectorized order N symmetric tensors, i.e.,

$$\mathbf{c} \in \mathrm{null}(\mathbf{Q}) \cap \mathrm{vec}(\mathrm{Sym}_R^d).$$

Overview

Algebraic algorithm outline

$$\underbrace{\mathbf{X}}_{\text{Known}} = \underbrace{\mathbf{C}}_{\text{Unknown}} \underbrace{(\mathbf{A} \odot \mathbf{B})^T}_{\text{Unknown}}$$

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until dim $\mathcal{E}_{h+1} = R^{h+1}$ and go to step 1.2.1;

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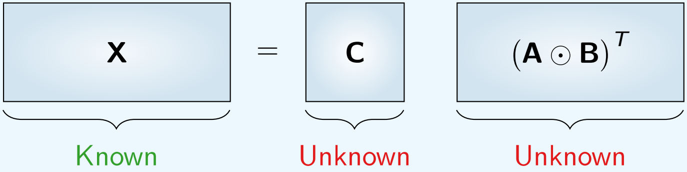

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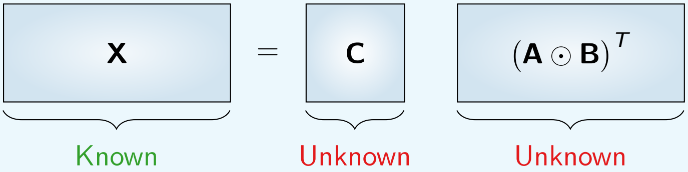

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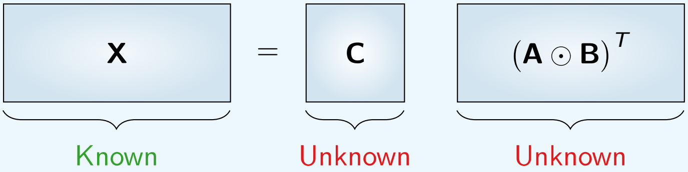
Algebraic algorithm outline



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Overview

Numerical and computational obstacles

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Work in progress

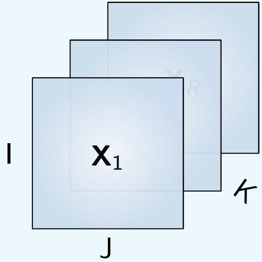
Advice is welcome!

Overview

Q for CPD

[Domanov and De Lathauwer 2013]

Let $\mathcal{X}$ be an order-3 tensor of dimension $(I \times J \times R)$, then



Definition: **Q** is a $C_I^2 C_J^2 \times C_{R+1}^2$ matrix whose k-th column can be written as

$$\mathbf{Q}(\cdot, k) = \text{vec}\left(\mathcal{C}_2(\mathbf{X}_{k_1} + \mathbf{X}_{k_2}) - \mathcal{C}_2(\mathbf{X}_{k_1}) - \mathcal{C}_2(\mathbf{X}_{k_2})\right)$$

where

- (k_1, k_2) is the k-th element of $\mathcal{Q}_R^2 = \{(k_1, k_2) : 1 \leq k_1 \leq k_2 \leq R\}$;
- C_N^2 is the binomial of N over 2;
- $\mathcal{C}_2(\mathbf{X}_h)$ is the matrix with the determinants of every 2×2 minors of $\mathbf{X}_h$.

Example

Let $\mathcal{X}$ be an order-3 tensor of dimension $(I \times 3 \times 2)$ such that

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{X}_2 = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}.$$

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All options present limitations!

Computational complexity

Let $\mathcal{X}$ be a $(I \times J \times R)$ tensor of rank R

Direct construction: the matrix $\mathbf{Q}$ has size $(C_I^2 C_J^2 \times C_{R+1}^2)$

- each entry requires two multiplications;
- the total number of entries is $(IJR(I-1)(J-1)(R+1))/8$;

assuming that $R \sim IJ$, then the computational complexity is $R^4/4$.

Gram construction: the matrix $\mathbf{Q}^T\mathbf{Q}$ has size $(C_{R+1}^2 \times C_{R+1}^2)$

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- $\text{tr}(\mathbf{A}_{h_i,k_j}^T \mathbf{A}_{k_\ell,h_m}) = \mathbf{a}_{h_i,k_j}^T \mathbf{a}_{k_\ell,h_m}$;
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Structure highlight

Example

Let $\mathcal{X}$ be an order-3 tensor of dimension $(I \times 3 \times 2)$ such that

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{X}_2 = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}.$$

The matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix} 2\alpha_1 & \gamma_1 - (\alpha_1 + \beta_1) & 2\beta_1 \\ 2\alpha_2 & \gamma_2 - (\alpha_2 + \beta_2) & 2\beta_2 \\ 2\alpha_3 & \gamma_3 - (\alpha_3 + \beta_3) & 2\beta_3 \end{bmatrix}.$$

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We can remark that

- the entries of α_1 appears† in $\mathbf{a}_1\mathbf{a}_2^T - \mathbf{a}_2\mathbf{a}_1^T$
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Exterior algebra and $\mathcal{C}_2$

Definition: The exterior product $\wedge : \mathbb{R}^I \times \mathbb{R}^J \to \mathbb{R}^{I \times J}$ is defined as

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Let $\mathbf{a}^T = \begin{bmatrix} a & b & c \end{bmatrix}$ and $\mathbf{b}^T = \begin{bmatrix} d & e & f \end{bmatrix}$ be two vectors, then

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- the elements highlighted are the entries of $\mathcal{C}_2(\begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix})$.

Gram matrix formulation

Property: $$\left\langle \mathrm{vec}(\mathcal{C}_2([\mathbf{a} \quad \mathbf{b}])), \mathrm{vec}(\mathcal{C}_2([\mathbf{a} \quad \mathbf{b}])) \right\rangle = 2\left\langle \mathrm{vec}(\mathbf{a} \wedge \mathbf{b}), \mathrm{vec}(\mathbf{a} \wedge \mathbf{b}) \right\rangle$$
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where $\mathbf{A}_{h_i k_j} = \mathbf{X}_{h_i}^T \mathbf{X}_{k_j}$ for $h_i \leq k_j$.

Computational complexity

Let $\mathcal{X}$ be a $(I \times J \times R)$ tensor of rank R

Direct construction: the matrix $\mathbf{Q}$ has size $(C_I^2 C_J^2 \times C_{R+1}^2)$

- each entry requires two multiplications;
- the total number of entries is $\big(IJR(I-1)(J-1)(R+1)\big)/8$;

assuming that $R \sim IJ$, then the computational complexity is $R^4/4$.

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Size	$\mathbf{Q}$		$\mathbf{Q}^T\mathbf{Q}$
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Table: Time estimates in seconds for computing $\mathbf{Q}$ and its Gram matrix.

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Possible new direction

Combining the two approaches

constructing $\mathbf{Q}$ benefiting from $\mathbf{Q}$ structure

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develop† a more convenient matrix-vector product, using $\mathbf{Q}$ structure.

† possibly based on the exterior product

Overview

Intersection shrinking

Once $\mathbf{Q}^{\dagger}$ is computed, the intersection subspace

$$\mathcal{E}_0 = \mathrm{null}(\mathbf{Q}) \cap \mathrm{vec}(\mathrm{Sym}_R^N)$$

has to be computed, cheking its dimension:

- if $\dim \mathcal{E}_0 = R$, then the algorithm proceed;
- if $\dim \mathcal{E}_0 > R$, then the intersection is shrinked, computing $\mathcal{E}_{h+1}$ such that

$$\mathcal{E}_{h+1} = (\mathbb{K}^R \otimes \mathcal{E}_h) \cap \mathrm{vec}(\mathrm{Sym}_R^{N+h})$$

  until $\dim \mathcal{E}_{h+1} = R^{h+1}$ and then proceed.

† or its Gram matrix $\mathbf{Q}^T\mathbf{Q}$.

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   as $\mathbf{E}_0 = \mathbf{G}_0\mathbf{F}_0$ where the matrix $\mathbf{G}_0$ has size $(R^2 \times N)$ and comes from the symmetrization of the canonical basis of order-2 tensors of size $(R \times R)$;

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   5.1 if $\hat{R}$ is equal to R, then reshape $\mathbf{E}_0$ as a $(R \times R \times R)$ and compute its CPD;

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   5.1 if $\hat{R}$ is equal to R, then reshape $\mathbf{E}_0$ as a $(R \times R \times R)$ and compute its CPD;
   5.2 if $\hat{R}$ is *not* equal to R, then shrink the intersection until $\hat{R} = R^{h+1}$.

$\mathcal{E}_{h+1}$ basis

If the matrix $\mathbf{E}_0$ of size $(R^2 \times \hat{R})$ span the intersection subspace $\mathcal{E}_0$ with $\hat{R} > R$ until $\hat{R}$ is equal to R^{h+1}, set $h = 0$ and repeat

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1. compute $\mathbf{F}_{h+1}$ the basis of
$$\mathcal{F}_{h+1} = \mathbb{K}^R \otimes \mathcal{E}_h$$
by a Kronecker product between the identity of order R and $\mathbf{E}_h$ the basis of $\mathcal{E}_h$;
2. compute $\mathbf{E}_{h+1}$ the basis of
$$\mathcal{E}_{h+1} = \mathcal{F}_{h+1} \cap \mathrm{vec}\left(\mathrm{Sym}_R^{(h+1)+2}\right)$$
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by symmetrizing the columns of $\mathbf{F}_{h+1}$;
3. orthogonalize $\mathbf{E}_{h+1}$;
4. check how many of $\mathbf{E}_{h+1}$ columns satisfy the symmetric conditions, updating $\hat{R}$.

Possible tool

Currently to avoid forming $\mathbf{F}_{h+1}$ and the symmetrizer matrix $\mathbf{G}_{h+1}$

1. compute $\mathbf{C}_{h+1}$ matrix of size $(R\hat{R} \times C^{h+2}_{R+2+h})$ carrying their product information;

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and this steps are repeated until R^{h+1} and $\hat{R}$ coincide.

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Study theoretically the enlargement-shrinking process to estimate m a priori and orthogonalize just once[†]

† or design an heuristic method

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4. reshape $\mathbf{R}$ as a tensor $\mathcal{R}$ of size $(R \times R \times R)$;
5. compute the CPD of $\mathcal{R}$ getting the factor $\mathbf{C}$ of size $(R \times R)$.

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- currently use Jenrich's algorithm, based on `GEVD`[†];

[†] problematic numerical stability Beltrán, Breiding, et al. 2019.

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- currently use Jenrich's algorithm, based on `GEVD`[†];
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- try to preserve (partially)-symmetric properties and tune CPD algorithm.

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- estimate the order of the symmetric tensor to product an auxiliary tensor of proper size;
- using the (partially)-symmetric structure into CPD of the auxiliary tensor.

Thank you for the attention!
Questions? Advice?

References I

- Beltrán, Carlos, Paul Breiding, and Nick Vannieuwenhoven (Jan. 2019). "Pencil-Based Algorithms for Tensor Rank Decomposition are not Stable". In: *SIAM J. Matrix Anal. Appl.* 40.2, pp. 739–773.
- Comon, P. and C. Jutten (2009). *Handbook of blind source separation: Independent component analysis and applications*. Academic press.
- Domanov, I. and L. De Lathauwer (July 2013). "On the uniqueness of the canonical polyadic decomposition of third-order tensors — Part I: Basic results and uniqueness of one factor matrix". In: *SIAM J. Matrix Anal. Appl.* 34.3, pp. 855–875.
- — (June 2016). "Generic Uniqueness of a Structured Matrix Factorization and Applications in Blind Source Separation". In: *IEEE J. Sel. Topics Signal Process.* 10.4, pp. 701–711.

References II

Evert E. Vandecappelle M., De Lathauwer L. (2020). *A recursive eigenspace computation for the canonical polyadic decomposition*. Tech. rep. 20-80. Leuven, Belgium: ESAT-STADIUS, KU Leuven.