# An algebraic algorithm for blind source separation and tensor decomposition

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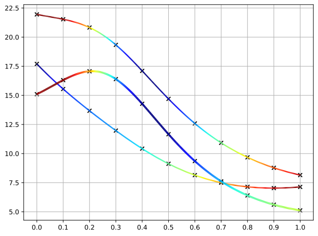
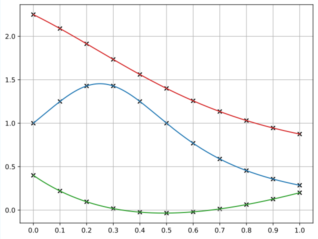
# Overview

# Blind Source Separation problem




$$= \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \times$$

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$$\underbrace{\mathbf{X}}_{\text{Known}} = \underbrace{\mathbf{M}}_{\text{Unknown}} \underbrace{\mathbf{S}^{T}}_{\text{Unknown}}$$

# Overview

# Constraints for uniqueness

Definition: A *deterministic* condition on $\mathbf{X}$ imposes a particular property of $\mathbf{X}$ that is always true.

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Definition: A *generic* condition on $\mathbf{X}$ depending on a parameter $\mathbf{z} \in \Omega$ holds almost everywhere, i.e., if the condition doesn't hold for $\mathbf{z} \in \Sigma \subset \Omega$, then $\mu(\Sigma) = 0$ with $\mu$ a measure absolute continuous w.r.t. the Lebesgue one.

# Deterministic conditions

- Statistical independence → Independent Component Analysis

$$\mathbf{X} = \mathbf{M} \underbrace{\mathbf{S}^T}_{\text{Ind}}$$

- Nonnegativity → Nonnegative Matrix Factorization

$$\mathbf{X} = \underbrace{\mathbf{M}}_{+} \underbrace{\mathbf{S}^T}_{+}$$

- Sparsity → Sparse Component Analysis

$$\mathbf{X} = \mathbf{M} \underbrace{\mathbf{S}^T}_{\text{Max0}}$$

- ...

[Comon and Jutten 2009]

# General case

$$\mathbf{X} = \mathbf{M} \; \mathbf{S}^T$$

$$\underbrace{\mathbf{X}}_{\mathbf{X}} = \underbrace{\mathbf{M} \; \mathbf{A}}_{\mathbf{M}_1} \; \underbrace{\mathbf{A}^{-1} \; \mathbf{S}^T}_{\mathbf{S}_1^T}$$

Uniqueness isn't guaranteed!

# Overview

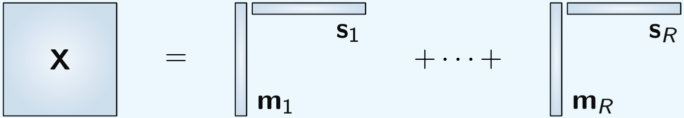
- The Blind Source Separation
  - Deterministic uniqueness
  - Generic uniqueness
- The Canonical Polyadic Decomposition
  - From the theorem to the algorithm
  - The bottleneck
- Algorithm improvements
  - Exterior algebra
  - Numerical results

## Constraints for uniqueness

Definition: A *deterministic* condition on $\mathbf{X}$ imposes a particular property of $\mathbf{X}$ that is always true.

Definition: A *generic* condition on $\mathbf{X}$ depending on a parameter $\mathbf{z} \in \Omega$ holds almost everywhere, i.e., if the condition doesn't hold for $\mathbf{z} \in \Sigma \subset \Omega$, then $\mu(\Sigma) = 0$ with $\mu$ a measure absolute continuous w.r.t. the Lebesgue one.

# Problem statement




$$\mathbf{X} = \mathbf{M}(\mathbf{z})\mathbf{S}^T(\mathbf{z}) = \sum_{r=1}^{R} \mathbf{m}_r(\mathbf{z}) \otimes \mathbf{s}(\boldsymbol{\xi}_r)$$

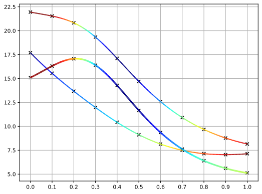
where

- $\mathbf{z} \in \Omega$ a subset of $\mathbb{R}^n$
- $\mathbf{m}_r(\mathbf{z})$ are linearly independent
- each $\mathbf{s}_r(\mathbf{z})$ depends on $\ell$ independent parameters, entries of $\boldsymbol{\xi}_r \in \mathbb{R}^\ell$
- each $\mathbf{s}_r(\mathbf{z}) = \mathbf{s}_r(\boldsymbol{\xi}_r)$ has the structure

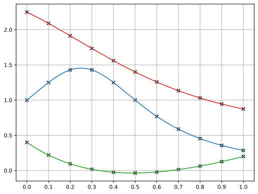
$$\mathbf{s}(\boldsymbol{\xi}_r) = \left[ \frac{p_1}{q_1} \circ \mathbf{f}(\boldsymbol{\xi}_r) \quad \dots \quad \frac{p_N}{q_N} \circ \mathbf{f}(\boldsymbol{\xi}_r) \right]$$

with $p_h, q_h$ polynomials and $\mathbf{f} = [f_1, \dots, f_\ell]$ a vectorial function.

# Example




$$= \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \times$$

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Given the observed mixtures, we assume that

- the mixture matrix $\mathbf{M}$ is constant and full rank;
- the source signals can be modeled by rational functions, i.e., the columns of $\mathbf{S}$ are sampled of

$$s(t) = \frac{a_0 + a_1 t + \cdots + a_p t^p}{b_0 + b_1 t + \cdots + b_q t^q} \qquad \text{with} \qquad a_i, b_i \in \mathbb{R}, \quad t \in [t_b, t_e]$$

- $\boldsymbol{\xi} = [a_0, \ldots, a_p, b_0, \ldots, b_q]$
- $\ell = p + q + 2$
- $\mathbf{f}$ is the identity

## A cookbook recipe for generic uniqueness [Domanov and De Lathauwer 2016]

Let $\mathbf{t}(\mathbf{x}) = \begin{bmatrix} \frac{p_1}{q_1}(\mathbf{x}) & \dots & \frac{p_N}{q_N}(\mathbf{x}) \end{bmatrix}^T$ for $\mathbf{x} \in \Theta = \{\mathbf{x} \in \mathbb{C}^\ell : q_1(\mathbf{x}) \cdots q_N(\mathbf{x}) \neq 0\}$, if

1. $\text{rank}\mathbf{M}(z) = R$ for a generic choice of $\mathbf{z}$
2. each $f_h$ is the ratio of two analytical functions on $\mathbb{C}^\ell$
3. there exists $\boldsymbol{\xi}_0 \in \mathbb{C}^\ell$ s.t. $\det\mathbf{J}(\mathbf{f}, \boldsymbol{\xi}_0) \neq 0$
4. the dimension of the span of $\mathbf{t}(\mathbf{x})$ for $x \in \Theta$ is at least $\hat{N}$
5. $\text{rank}\mathbf{J}(\mathbf{t}, \mathbf{x}) > \hat{\ell}$ for a generic choice of $\mathbf{z}$
6. $R \leq \hat{N} - \hat{\ell}$

then

$$\mathbf{X} = \sum_{r=1}^{R} \mathbf{m}_r(\mathbf{z}) \otimes \mathbf{s}(\boldsymbol{\xi}_r)$$

is generically unique.

# Remarks for BSS

It is assumed that the columns of $\mathbf{S}$ are values of the rational function

$$\mathbf{t} : \mathbf{x} \rightarrow \begin{bmatrix} \frac{p_1}{q_1}(\mathbf{x}) & \dots & \frac{p_N}{q_N}(\mathbf{x}) \end{bmatrix}^T .$$

The columns of $\mathbf{S}$ belong to an algebraic variety $\mathcal{V}$ which is described by a finite system of polynomials $\{P_k\}_{k=1}^K$

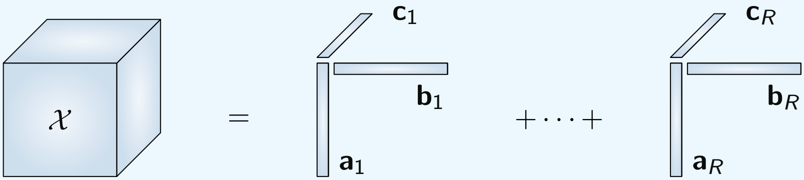
$$\mathcal{V} = \Big\{ (z_1, \dots, z_N) \in \mathbb{C}^N : P_k(z_1, \dots, z_N) = 0 \Big\}.$$

Composed with the function $\mathbf{f}$

# Overview

# Link with the CPD

$$\mathcal{X} = \mathbf{a}_1 \otimes \mathbf{b}_1 \otimes \mathbf{c}_1 + \ldots + \mathbf{a}_R \otimes \mathbf{b}_R \otimes \mathbf{c}_R$$

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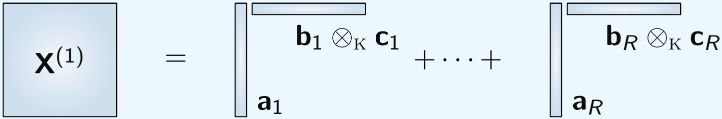
$$\mathbf{X}^{(1)} = \mathbf{a}_1 \otimes (\mathbf{b}_1 \otimes_{\mathrm{K}} \mathbf{c}_1)^T + \ldots + \mathbf{a}_R \otimes (\mathbf{b}_R \otimes_{\mathrm{K}} \mathbf{c}_R)^T$$




$$(\mathbf{b}_r \otimes_{\mathrm{K}} \mathbf{c}_r) \in \mathcal{V} = \left\{ \mathrm{vec}(\mathbf{Z}) : \begin{vmatrix} z_{i_1 j_1} & z_{i_1 j_2} \\ z_{i_2 j_1} & z_{i_2 j_2} \end{vmatrix} = 0 \right\}$$

## Overview

# Algebraic algorithm outline

$$\underbrace{\mathbf{X}}_{\text{Known}} = \underbrace{\mathbf{M}}_{\text{Unknown}} \underbrace{\mathbf{S}^T}_{\text{Unknown}}$$

1. compute $\mathbf{M}^{-1}$ from $\mathbf{X}$;
2. compute $\mathbf{S}$ as $\mathbf{M}^{-1}\mathbf{X}$ transposed

# Equivalent condition I

$\mathbf{a}$ is a column of $\mathbf{M}^{-1}$ if and only if $\mathbf{X}^T\mathbf{a}$ is equal to a column of $\mathbf{S}$

$$\Downarrow$$

$$\mathbf{X}^T\mathbf{a} = (\mathbf{x}_1^T\mathbf{a}, \ldots, \mathbf{x}_N^T\mathbf{a}) = (z_1, \ldots, z_N) \in \mathcal{V}$$

$$\Updownarrow$$

$$P_k(\mathbf{x}_1^T\mathbf{a}, \ldots, \mathbf{x}_N^T\mathbf{a}) = 0 \text{ for } k = 1, \ldots, K$$

$$\Updownarrow$$

$$P_k^{\otimes}(\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)(\mathbf{a} \otimes \cdots \otimes \mathbf{a}) = 0 \text{ for } k = 1, \ldots, K$$

where $P_k^{\otimes}(\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)$ is the vector obtained by formal substitution of $(z_1, \ldots, z_N)$ by $\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T$ and the scalar multiplication by the tensor product.

## Equivalent condition II

**a** is a column of $\mathbf{M}^{-1}$ if and only if $\mathbf{X}^T\mathbf{a}$ is equal to a column of $\mathbf{S}$

$$\Downarrow$$

$$P_k^{\otimes}(\mathbf{x}_1^T,\ldots,\mathbf{x}_N^T)(\mathbf{a}\otimes\cdots\otimes\mathbf{a}) = 0 \text{ for } k = 1,\ldots,K$$

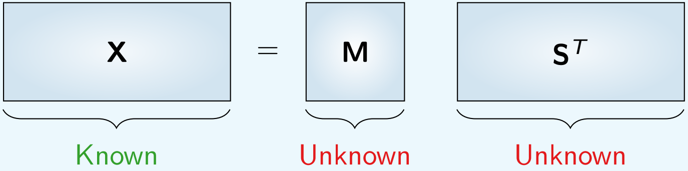
$$\Updownarrow$$

$$\mathbf{Q}\mathrm{vec}(\mathbf{a}^{\otimes p}) = \begin{bmatrix} P_1^{\otimes}(\mathbf{x}_1^T,\ldots,\mathbf{x}_N^T) \\ \vdots \\ P_K^{\otimes}(\mathbf{x}_1^T,\ldots,\mathbf{x}_N^T) \end{bmatrix} \mathrm{vec}(\mathbf{a}\otimes\cdots\otimes\mathbf{a}) = 0$$

$$\Updownarrow$$

The columns of $\mathbf{M}^{-1}$ belong to the intersection of $\mathbf{Q}$ kernel and the subspace of vectorized order $p$ symmetric tensors.

# Algebraic algorithm outline



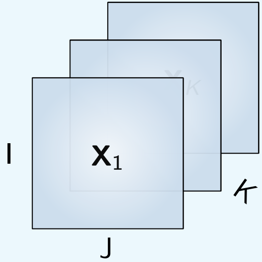

1. compute $\mathbf{M}^{-1}$ from $\mathbf{X}$;
   1.1 compute $\mathbf{Q}$;
   1.2 compute $\text{null}(\mathbf{Q})$ intersected with the space of vectorized symmetric tensors;
2. compute $\mathbf{S}$ as $\mathbf{M}^{-1}\mathbf{X}$ transposed.

# Overview

## $\mathbf{Q}$ for CPD

[Domanov and De Lathauwer 2013]

Let $\mathcal{X}$ be an order-3 tensor of dimension $(I \times J \times K)$, then




Definition: $\mathbf{Q}$ is a $C_I^2 C_J^2 \times C_{K+1}^2$ matrix whose $k$-th column can be written as

$$\mathbf{Q}(\cdot, k) = \text{vec}(\mathcal{C}_2(\mathbf{X}_{k_1} + \mathbf{X}_{k_2}) - \mathcal{C}_2(\mathbf{X}_{k_1}) - \mathcal{C}_2(\mathbf{X}_{k_2}))$$

where

- $(k_1, k_2)$ is the $k$-th element of $\mathcal{Q}_K^2 = \{(k_1, k_2) : 1 \leq k_1 \leq k_2 \leq K\}$;
- $C_N^2$ is the binomial of $N$ over 2;
- $\mathcal{C}_2(\mathbf{X}_h)$ is the matrix with the determinants of every $2 \times 2$ minors of $\mathbf{X}_h$.

## Example

Let $\mathcal{X}$ be an order-3 tensor of dimension $(I \times 3 \times 2)$ such that

$$\mathbf{X}_1 = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} \quad \text{and} \quad \mathbf{X}_2 = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix}$$

The matrix $Q = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{bmatrix}$ associated with $\mathcal{X}$ slices is such that

- $\mathbf{q}_1 = \text{vec}(\mathcal{C}_2(\mathbf{X}_1 + \mathbf{X}_1) - \mathcal{C}_2(\mathbf{X}_1) - \mathcal{C}_2(\mathbf{X}_1)) = 2\text{vec}(\mathcal{C}_2(\mathbf{X}_1))$
- $\mathbf{q}_2 = \text{vec}(\mathcal{C}_2(\mathbf{X}_1 + \mathbf{X}_2) - \mathcal{C}_2(\mathbf{X}_1) - \mathcal{C}_2(\mathbf{X}_2))$
- $\mathbf{q}_3 = \text{vec}(\mathcal{C}_2(\mathbf{X}_2 + \mathbf{X}_2) - \mathcal{C}_2(\mathbf{X}_2) - \mathcal{C}_2(\mathbf{X}_2)) = 2\text{vec}(\mathcal{C}_2(\mathbf{X}_2))$

The compound matrices are

- $\mathcal{C}_2(\mathbf{X}_1) = \begin{bmatrix}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \boldsymbol{\alpha}_3\end{bmatrix}$
- $\mathcal{C}_2(\mathbf{X}_2) = \begin{bmatrix}\boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \boldsymbol{\beta}_3\end{bmatrix}$
- $\mathcal{C}_2(\mathbf{X}_1 + \mathbf{X}_2) = \begin{bmatrix}\boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 & \boldsymbol{\gamma}_3\end{bmatrix}$

$$\Rightarrow \quad \mathbf{Q} = \begin{bmatrix} 2\boldsymbol{\alpha}_1 & \boldsymbol{\gamma}_1 - (\boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1) & 2\boldsymbol{\beta}_1 \\ 2\boldsymbol{\alpha}_2 & \boldsymbol{\gamma}_2 - (\boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2) & 2\boldsymbol{\beta}_2 \\ 2\boldsymbol{\alpha}_3 & \boldsymbol{\gamma}_3 - (\boldsymbol{\alpha}_3 + \boldsymbol{\beta}_3) & 2\boldsymbol{\beta}_3 \end{bmatrix}$$

# Overview

## Exterior algebra and $\mathcal{C}_2$

**Definition:** The exterior product $\wedge : \mathbb{R}^I \times \mathbb{R}^J \to \mathbb{R}^{I \times J}$ is defined as

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T.$$

### Example

Let $\mathbf{a}^T = \begin{bmatrix} a & b & c \end{bmatrix}$ and $\mathbf{b}^T = \begin{bmatrix} d & e & f \end{bmatrix}$ be two vectors, then

$$\mathbf{a} \wedge \mathbf{b} = \begin{bmatrix} 0 & ae - bd & af - cd \\ -ae + bd & 0 & bf - ce \\ -af + cd & -bf + ce & 0 \end{bmatrix}$$

The matrix $\mathbf{a} \wedge \mathbf{b}$ has the properties:

- the diagonal entries are zeros;
- it is skew-symmetric;
- the elements highlighted are the entries of $\mathcal{C}_2([\mathbf{a} \quad \mathbf{b}])$;

# Algebraic algorithm optimization steps

$$\text{Property: } \left\langle \text{vec}(\mathcal{C}_2([\mathbf{a} \quad \mathbf{b}])), \text{vec}(\mathcal{C}_2([\mathbf{a} \quad \mathbf{b}])) \right\rangle = 2\left\langle \text{vec}(\mathbf{a} \wedge \mathbf{b}), \text{vec}(\mathbf{a} \wedge \mathbf{b}) \right\rangle = 4||\mathbf{a}||^2||\mathbf{b}||^2 - \langle \mathbf{a}, \mathbf{b} \rangle^2$$

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Idea: Using the exterior product to improve the algebraic algorithm.

- $\mathbf{Q}$ is not explicitly constructed
- $\text{null}(\mathbf{Q}) = \text{null}(\mathbf{Q}^T\mathbf{Q})$
- $\langle \mathbf{q}_k, \mathbf{q}_h \rangle = \langle \text{vec}(\mathcal{C}_2(\mathbf{Y})), \text{vec}(\mathcal{C}_2(\mathbf{Z})) \rangle$ with $\mathbf{Y}, \mathbf{Z}$ slices or a linear combination of slices of the input tensor $\mathcal{X}$

⇓

  - Pre-compute $\mathbf{A}_{h_i k_j} = \mathbf{X}_{h_i}^T \mathbf{X}_{k_j}$ for $h_i \leq k_j$
  - Unified formula for $\mathbf{Q}^T\mathbf{Q}$

$$\langle \mathbf{q}_k, \mathbf{q}_h \rangle = \text{tr}(\mathbf{A}_{h_1 k_1})\text{tr}(\mathbf{A}_{h_2 k_2}) + \text{tr}(\mathbf{A}_{h_1 k_2})\text{tr}(\mathbf{A}_{h_2 k_1}) - \text{tr}(\mathbf{A}_{h_1 k_1}^T \mathbf{A}_{h_1 k_2}) - \text{tr}(\mathbf{A}_{h_1 k_2}^T \mathbf{A}_{h_2 k_1})$$

# Overview

# Preliminary numerical results (work in progress)

Given an input random tensor $\mathcal{X}$ of dimensions $(I \times J \times K)$ such that its rank is

$$R = I = (J-1)(K-1)$$

| dimensions | rank | *naive* | | *smart* | |
|---|---|---|---|---|---|
| | | **CPU time** | **err** | **CPU time** | **err** |
| $20 \times 5 \times 6$ | $R = 20$ | 1.089e−01s | 5.518e−13 | 8.353e−03s | 5.009e−13 |
| $24 \times 5 \times 7$ | $R = 24$ | 7.600e−02s | 9.949e−08 | 4.408e−03s | 1.026e−07 |
| $25 \times 6 \times 6$ | $R = 25$ | 8.977e−02s | 4.656e−07 | 5.223e−03s | 5.689e−07 |
| $28 \times 5 \times 8$ | $R = 28$ | 1.400e−01s | 1.849e−11 | 9.695e−03s | 1.601e−11 |
| $30 \times 6 \times 7$ | $R = 30$ | 3.538e−01s | 2.214e−11 | 2.236e−02s | 8.866e−12 |

The link between rank and dimensions is meant to satisfy [Domanov and De Lathauwer 2016] theorem.

# Conclusive remarks

- conditions that guarantee generic uniqueness;
- from the theorem structure to a CPD algorithm;
- bottleneck due to compound matrices;
- optimization based on the exterior product.

Thank you for the attention!
Questions?

# References I


- Comon, P. and C. Jutten (2009). *Handbook of blind source separation: Independent component analysis and applications*. Academic press.
- Domanov, I. and L. De Lathauwer (July 2013). “On the uniqueness of the canonical polyadic decomposition of third-order tensors — Part I: Basic results and uniqueness of one factor matrix”. In: *SIAM J. Matrix Anal. Appl.* 34.3, pp. 855–875.
- — (June 2016). “Generic Uniqueness of a Structured Matrix Factorization and Applications in Blind Source Separation”. In: *IEEE J. Sel. Topics Signal Process.* 10.4, pp. 701–711.