UNIVERSITÀ DEGLI STUDI DI TRENTO

HOSVD FOR MULTISPECTRAL IMAGES

A numerical approach to plant biodiversity estimation

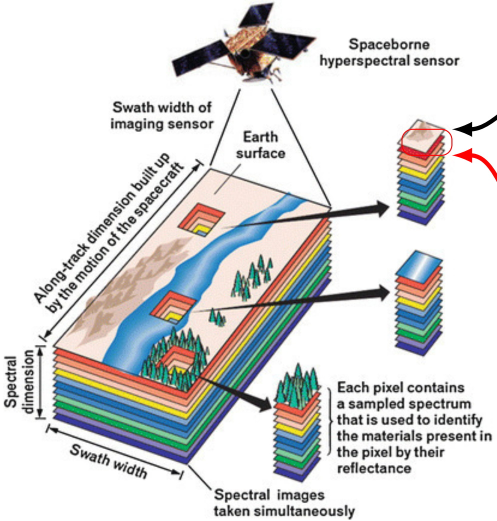
Supervisor:
prof. Alessandra Bernardi
Co-advisor:
prof. Duccio Rocchini

Candidate:
Martina Iannacito

University of Trento, July 17, 2019

Contents

Remote sensing



NIR, $\sim 750 - 2500\,\mathrm{nm}$

RED, $\sim 620 - 750\,\mathrm{nm}$

Image source [Bedini, 2017]

Vegetation indexes

Normalized Difference Vegetation Index

Given a region R, let $RED, NIR \in \mathbb{M}^{m\times n}(\mathbb{R})$ be respectively the RED and the NIR raster band of R imagery. The normalized difference vegetation index of region R is $NDVI \in \mathbb{M}^{m\times n}(\mathbb{R})$ such that

$$\text{NDVI}_{ij} = \frac{NIR_{ij} - RED_{ij}}{NIR_{ij} + RED_{ij}}$$

for every $i \in \{1, \dots, m\}$ and for every $j \in \{1, \dots, n\}$, when it is defined.

Biodiversity indexes

Given a spectral image of a sample area, let N be the image radiometric resolution and let p_i be the relative abundances of the i-th value for every $i \in \{1, \dots, N\}$.

Rényi index

$$I_R = -\log \sum_{i=1}^{N} p_i^2.$$

Fixed a distance function d, we build up a pairwise spectral difference matrix $D \in \mathbb{M}^N(\mathbb{R})$ such that $D_{ij} = d(i, j)$ for every $i, j \in \{1, \dots, N\}$.

Rao's Q index

$$I_{RQ} = \sum_{j=1}^{N} \sum_{i=1}^{N} p_i p_j D_{ij}.$$

The problem

Example

A band of Earth's surface from the MODIS sensor with a low spectral resolution, 5600 m, needs around 99 MB of storage memory.

Mathematical compression techniques.

Matrices	Tensors
SVD	HOSVD

Singular values decomposition

Singular values

Given a matrix $A \in \mathbb{M}^{M \times N}(\mathbb{R})$ and λ_i the eigenvalues of $A^T A$, the singular values of A are $\sigma_i = \sqrt{\lambda_i}$ for every $i \in \{1, \ldots, N\}$

Theorem

Given a matrix $A \in \mathbb{M}^{M \times N}(\mathbb{R})$ of rank r with singular values σ_i for $i \in \{1, \ldots, r\}$ such that $\sigma_i \leq \sigma_{i+1}$, then exist U,V and Σ matrix such that:

I $U \in$ O(M), $V \in$ O(N), $\Sigma \in \mathbb{M}^{M \times N}(\mathbb{R})$;

II $\Sigma_{i,i} = \sigma_i$ for $i \in \{1, \ldots, r\}$;

III $\Sigma_{i,j} = 0$ otherwise;

IV $A = U \Sigma V^T$.

Singular values application

Eckart-Young theorem

Given a matrix $A \in \mathbb{M}^{M\times N}(\mathbb{R})$ of rank r with singular values σ_i for every $i \in \{1,\ldots,r\}$ such that $\sigma_i \leq \sigma_{i+1}$. If for every $i \in \{1,\ldots,r\}$ it is defined the diagonal matrix $S_i \in \mathbb{M}^{M\times N}(\mathbb{R})$ such that $(S_i)_{i,i} = \sigma_i$ and $(S_i)_{i,j} = 0$ otherwise, then:

I $\left\| A - US_1V^T \right\| \leq \|A - X\|$ for every $X \in \mathbb{M}^{M\times N}(\mathbb{R})$ such that $rank(X) = 1$;

II for every $k \in \{1,\ldots,r\}$, we have

$$\left\| A - \sum_{i=1}^{k} US_iV^T \right\| \leq \|A - X\|$$

for every $X \in \mathbb{M}^{M\times N}(\mathbb{R})$ such that $rank(X) \leq k$

Photo application

(a) Original

(b) 1000

(c) 10

(d) 50

(e) 100

Is it possible to generalise to SVD to tensors?

Let $\mathcal{A} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$ be a tensor

The multilinear rank of $\mathcal{A}$ is the tuple $(r_1, \ldots, r_d)$ such that there exists a minimal separable tensor subspace $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_d$, containing $\mathcal{A}$, in the following sense:

$$r_i = \min_{\mathcal{V}_i \text{ subspace of } \mathbb{K}^{n_i}} \mathrm{dim}(\mathcal{V}_i)$$

for every $i \in \{1, \ldots, d\}$.

Low multi-linear rank approximation

Given a tuple $(r_1, \ldots, r_d)$, we might ask if there exists and how to find the tensor $\mathcal{M}$ such that

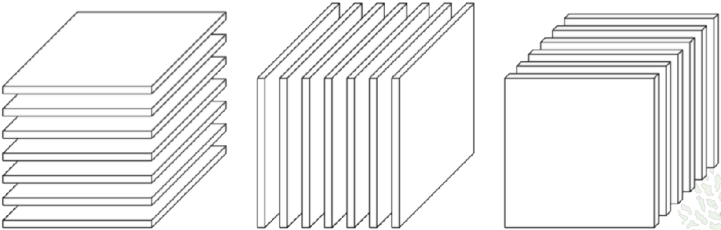
$$\mathcal{M} = \arg \inf_{mlrank(\mathcal{T}) \leq (r_1, \ldots, r_d)} \|\mathcal{A} - \mathcal{T}\| .$$

Flattening

Given $\mathcal{A} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$ a tensor, its k-flattening is $\mathcal{A}$ seen as element of $(\mathbb{K}^{n_k}) \otimes (\mathbb{K}^{n_1} \otimes \cdots \otimes \hat{\mathbb{K}}^{n_k} \otimes \cdots \otimes \mathbb{K}^{n_d})$ for every $k \in \{1, \ldots, d\}$.

Notation

$$\mathcal{A}_{(k)} \in (\mathbb{K}^{n_k}) \otimes (\mathbb{K}^{n_1} \otimes \cdots \otimes \hat{\mathbb{K}}^{n_k} \otimes \cdots \otimes \mathbb{K}^{n_d}).$$



Tucker's decomposition

Given a tensor $\mathcal{A} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$ such that $\mathcal{A} \in \mathscr{V}_1 \otimes \cdots \otimes \mathscr{V}_d$ where $\mathscr{V}_i \subseteq \mathbb{K}^{n_i}$ and $\dim(\mathscr{V}_i) = r_i$ for every $i \in \{1, \ldots, d\}$, then for every $i \in \{1, \ldots, d\}$ it can be chosen a matrix B_i such that:

I $\mathscr{V}_i =$span$\{(B_i)_{\cdot,k}\}_{k=1}^{r_i}$;

II $\mathcal{A} = (B_1, \ldots, B_d)\mathcal{C}$;

where $\mathcal{C} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$ is the core tensor.
Applying the Moore-Penrose pseudo-inverse, we get

$$\mathcal{C} = (B_1^{\dagger}, \ldots, B_d^{\dagger})\mathcal{A}$$

HOSVD

How to choose the basis

Remark

rank$(\mathcal{A}_i) = r_i =$dim$(\mathcal{V}_i)$ for every $i \in \{1, \ldots, D\}$.

For every $i \in \{1, \ldots, d\}$ we compute SVD of the flattening $\mathcal{A}_{(i)}$

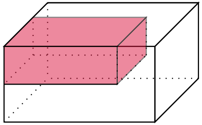
$$\mathcal{A}_{(i)} = U_i \Sigma_i (V_i)^T.$$

We set $(B_i)_{\cdot,k} = (U_i)_{\cdot,k}$ for every $k \in \{1, \ldots, r_i\}$ and for every $i \in \{1, \ldots, d\}$ and we prove that

$$\mathcal{A} = (U_1, \ldots, U_d)\mathcal{C} \quad \text{with} \quad \mathcal{C} = (U_1^H, \ldots, U_d^H)\mathcal{A}.$$

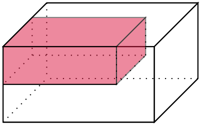
HOSVD implementation

Visually

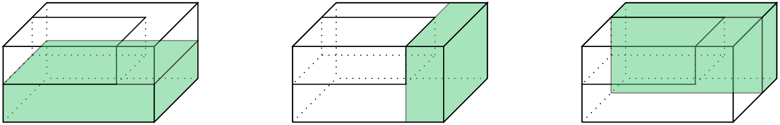


HOSVD implementation

Visually

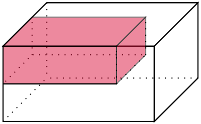


Truncation

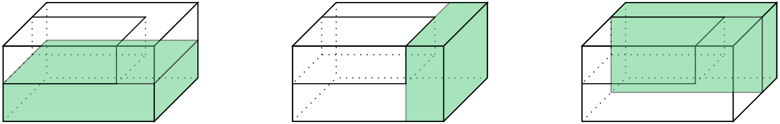


HOSVD implementation

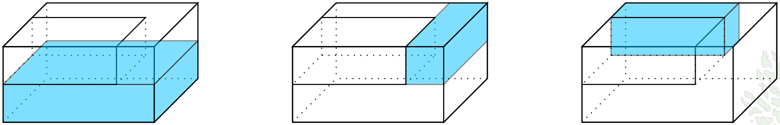
Visually



Truncation



Sequentially truncation



HOSVD

Recent versions

Algorithm 1 T-HOSVD

Input: a tensor $\mathcal{A} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$
Input: a target multilinear rank $(r_1, \ldots, r_d)$
Output: the T-HOSVD basis in matrix form $(\overline{U}_1, \ldots, \overline{U}_d)$
Output: the T-HOSVD core tensor $\mathcal{C} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$
for $i = 1, 2, \ldots, d$ **do**
    Compute SVD of $\mathcal{A}_{(i)}$, i.e. $\mathcal{A}_{(i)} = U_i \Sigma_i V_i^T$;
    Store in $\overline{U}_i$ the first r_i columns of U_i
end
$\mathcal{C} \leftarrow (\overline{U}_1^H, \ldots, \overline{U}_d^H)\mathcal{A}$;

[De Lathauwer, 2000], [Vannieuwenhoven, 2017]

HOSVD

Recent versions

Algorithm 2 ST-HOSVD

Input: a tensor $\mathcal{A} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$
Input: a target multilinear rank $(r_1, \ldots, r_d)$
Output: the ST-HOSVD basis in matrix form $\hat{U}_1, \ldots, \hat{U}_d$
Output: the ST-HOSVD core tensor $\mathcal{C} \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_d}$
$\mathcal{C} \leftarrow \mathcal{A}$;
for $i = 1, 2, \ldots, d$ **do**
  Compute SVD of $\mathcal{C}_{(i)}$, i.e. $\mathcal{C}_{(i)} = U_i \Sigma_i V_i^T$;
  Store in $\hat{U}_i$ the first r_i columns of U_i;
  $\mathcal{C} \leftarrow (\hat{U}_i^H)\cdot_i \mathcal{C}$;
end

[De Lathauwer, 2000], [Vannieuwenhoven, 2017]

Approach

Chosen Europe MOD13A3v006 and Earth MOD13C2v006 dataset, then:

I. Compute the indexes over NASA NDVIs;
II. Generating tensors with RED and NIR band for each element of both dataset;
III. Decompose and recompose tensor at target multilinear rank

$$(r, r, 2)$$

for every $r \in \{10, 50, 100, 500, 1000\}$ with ST-HOSVD and T-HOSVD;
IV. Compute NDVI image for each element of both dataset;
V. Compute both biodiversity indexes.
VI. Measure error between index over NASA NDVI and self-made NDVI.

Compression rates

Rank	Europe	Earth
	Rel	Rel
10	0.0019	0.0021
50	0.0095	0.0105
100	0.0191	0.0212
500	0.1024	0.1138
1000	0.2222	0.2469

Table: Rate of compression.

Rényi index

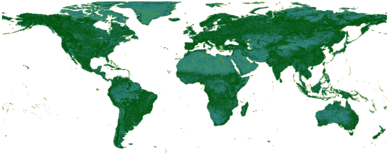
Table

	T-HOSVD				
Rank	10	50	100	500	1000
$\mathbb{E}[epO]$	0.1351	0.0915	0.084	0.0601	0.0556
$\mathbb{V}\text{ar}[epO]$	0.0001	0.0001	0.0002	0.0002	0.0001
$\min epO$	0.1119	0.0727	0.0638	0.0443	0.0424
$\max epO$	0.1564	0.1154	0.121	0.0963	0.0792

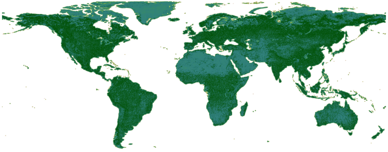
Table: Statistics for Rényi index over $N \in n_{N,W,T,j}$.

Rényi index

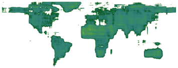
Best case



(f) Original



(g) 1000



(h) 10

(i) 50

(j) 100

(k) 500

Rao index

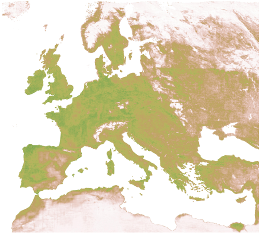
	ST-HOSVD				
Rank	10	50	100	500	1000
$\mathbb{E}[epO]$	0.6328	0.3604	0.293	0.2081	0.1951
$\mathbb{V}\mathrm{ar}[epO]$	0.0038	0.0011	0.0001	0.0003	0.0003
$\min epO$	0.4825	0.2724	0.2185	0.1326	0.1144
$\max epO$	0.7246	0.4065	0.3722	0.3871	0.3917

Table: Statistics for Rao index over $N \in n_{R,E,S,j}$.

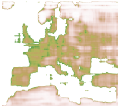
Rao index

Best case

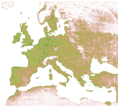
(m) Original



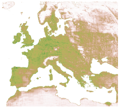
(n) 1000



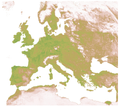
(o) 10



(p) 50



(q) 100



(r) 500

0 1 2

Wrap up the ideas

- High memory saving;
- extremely good results for Rényi index;
- appreciable results for Rao index.

Thanks for your attention

Questions?